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Equivalent Analogy of Mesoscopic RLC Circuit and Its Thermal Effect

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Abstract A new method of quantizing the mesoscopic RLC circuit is proposed, i.e., such a circuit can be equivalent to a changing mass simple harmonic oscillator. By virtue of the thermal entangled state representation and the generalized Hellmann-Feynman Theorem, the thermal effect for the system is investigated.

Keywords Thermal effect · Thermal entangled state · Generalized Hellmann-Feynman Theorem

1 Introduction

Due to the development of nanometer techniques and microelectronics, a substantial interest has revived the physical research on the mesoscopic circuits [1–5]. In recent years, the progress of quantum computation and quantum information, has made such a field become a hot spot again [6–11]. A single LC (inductance-capacitance) nondissipative mesoscopic circuit is a fundamental cell in mesoscopic circuits and its quantization and quantum effects were first discussed by Louisell [12], as is important for us to investigate complicated mesoscopic circuits. For the dissipation is inevitable in the practical mesoscopic circuits, the effect of R (resistance) on the circuits must be taken into account. To our knowledge, the

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representative quantization methods mainly include the following two cases: (1) by virtue of the quantization scheme of damping harmonic oscillators given by [13]; (2) Simulating RLC circuit as an electromagnetic harmonic oscillator coupled to a phonon bath [14]. Different from the above two cases, we think the introduction of R is equivalent to exerting a dissipative "force" $\mathcal{F}$ ($\mathcal{F}t$ is considered as generalized impulse) on the system and the dissipative energy of R should be attributed to the generalized kinetic energy. So we can follow Dirac's standard canonical quantization method [15], by which the derivation of Hamilton operator seems to be more natural.

2 Quantization of Mesoscopic RLC Circuit and Its Equivalent Analogy

For the mesoscopic RLC circuit (see Fig. 1), supposed that at some time t it is excited by an impulse source (the switch time $\tau \to 0$), if denoting the charge q as generalized coordinate, the potential energy of the system is

$$\mathcal{V} = \frac{q^2}{2C}, \tag{1}$$

and considering that the dissipative "force" applies negative work to the system, the kinetic energy corresponding to $\dot{q}$ is

$$\mathcal{T} = \frac{1}{2}L\dot{q}^2 - \dot{q}^2 Rt. \tag{2}$$

The Lagrangian function of the system is

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \frac{1}{2}L\dot{q}^2 - \dot{q}^2 Rt - \frac{q^2}{2C}. \tag{3}$$

Then the generalized momentum of the system can be obtained

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = L\dot{q} - 2\dot{q}Rt. \tag{4}$$

Note that the generalized momentum p is the explicit function of time, as is easy to understand. The introduction of the resistance is equivalent to exert a dissipative "force" on the system, which must lead to the momentum varying with time. Thus, the Hamiltonian of the system is

$$\mathcal{H} = p\dot{q} - \mathcal{L} = \frac{p^2}{2(L-2Rt)} + \frac{q^2}{2C}. \tag{5}$$

For p and q are a pair of canonical conjugate variables, according to Dirac's standard canonical quantization method, endowing the quantization condition

$$[\hat{q}, \hat{p}] = i\hbar, \tag{6}$$

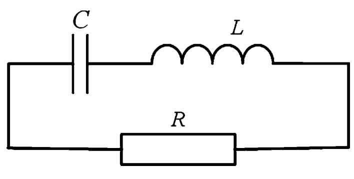

Fig. 1 Mesoscopic RLC circuit

the Hamilton operator is

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2L(t)} + \frac{\hat{q}^2}{2C}, \tag{7}$$

where $L(t) \equiv L - 2Rt$ is referred to as the equivalent inductance. Thus the mesoscopic RLC circuit can be regarded as a nondissipative changing inductance mesoscopic LC circuit, which is equivalent to a changing mass quantum simple harmonic oscillator, and the corresponding Hamilton operator is turned into

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2\mu} + \frac{1}{2}\mu\omega^2\hat{q}^2, \tag{8}$$

where $\mu \equiv L(t)$, $\omega = \sqrt{\frac{1}{L(t)C}}$. Introducing the following bosonic operators

$$\hat{a} = \frac{1}{\sqrt{2\mu\omega\hbar}}(\mu\omega\hat{q} + i\hat{p}), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2\mu\omega\hbar}}(\mu\omega\hat{q}_0 - i\hat{p}_0), \tag{9}$$

we have

$$\hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{10}$$

3 Quantum Fluctuations at Finite Temperature

For the dissipative circuits produce Joule heat themselves and are affected by the external environment, it is necessary to consider the effect of temperature on the circuit. In this section we shall employ the thermal entangled state representation [16] and the generalized Hellmann-Feynman Theorem [17] to investigate the thermal effect of the mesoscopic circuit.

3.1 By Virtue of Thermal Entangled State Representation

In order to convert the evaluations of ensemble averages at finite temperature into the equivalent expectation values with a pure state, Takahashi and Umezawa invented thermofield dynamics [18]. Every state $|n\rangle$ in the original real field space is accompanied by a corresponding fictitious state $|\tilde{n}\rangle$ in the fictitious space. A similar rule holds for operators: every operator $\hat{a}$ has an image $\widetilde{\hat{a}}$, which satisfies $[\hat{a}, \widetilde{\hat{a}}] = 0$ and $[\widetilde{\hat{a}}, \widetilde{\hat{a}}^\dagger] = 1$. For the harmonic oscillator system labeled by $\hat{\mathcal{H}}$, at finite temperature the vacuum state of the L-C circuit becomes the thermo vacuum state

$$|0(\beta)\rangle = \hat{s}(\theta)|0\tilde{0}\rangle = \exp\left[-\theta\left(\hat{a}\widetilde{\hat{a}} - \hat{a}^\dagger\widetilde{\hat{a}}^\dagger\right)\right]|0\tilde{0}\rangle, \tag{11}$$

where $\hat{s}(\theta)$ is named as thermo operator, and $\tanh\theta = \exp(-\frac{\beta\hbar\omega}{2})$, ($\beta = \frac{1}{kT}$, k is the Boltzmann constant). In order to discuss the problem conveniently, we introduce the thermo entangled state representation [16]. Displacing $|0(\beta)\rangle$ by the operator $D(\tau) = \exp(\tau\hat{a}^\dagger - \tau^*\hat{a})$ ($\tau = \tau_1 + i\tau_2$ is complex) leads to the thermo entangled state representation

$$|\tau\rangle = \exp\left[-\frac{1}{2}|\tau|^2 + \tau\hat{a}^\dagger - \tau^*\widetilde{\hat{a}}^\dagger + \hat{a}^\dagger\widetilde{\hat{a}}^\dagger\right]|0, \tilde{0}\rangle, \tag{12}$$

which satisfies the following relations

$$\hat{a}|\tau\rangle = \left(\tau + \tilde{\hat{a}}^{\dagger}\right)|\tau\rangle, \qquad \tilde{\hat{a}}|\tau\rangle = \left(-\tau^{*} + \hat{a}^{\dagger}\right)|\tau\rangle. \tag{13}$$

The complete set and orthogonality of $|\tau\rangle$ are, respectively,

$$\int \frac{d^2\tau}{\pi}|\tau\rangle\langle\tau| = 1, \qquad \langle\tau'|\tau\rangle = \pi\delta\left(\tau - \tau'\right)\delta\left(\tau^{*} - \tau'^{*}\right). \tag{14}$$

By using $|\tau\rangle$ the thermo squeezing operator is neatly expressed as

$$\hat{s}(\theta) = \int \frac{d^2\tau}{\pi\lambda}|\tau/\lambda\rangle\langle\tau|, \quad \lambda^2 = \frac{1+\tanh\theta}{1-\tanh\theta}, \tag{15}$$

thus

$$\hat{s}(\theta)|\tau\rangle = 1/\lambda|\tau/\lambda\rangle, \tag{16}$$

$$|0(\beta)\rangle = \int \frac{d^2\tau}{\pi\lambda}|\tau/\lambda\rangle\langle\tau|0,\tilde{0}\rangle = \int \frac{d^2\tau}{\pi\lambda}|\tau/\lambda\rangle e^{-\frac{1}{2}|\tau|^2}. \tag{17}$$

We shall employ $|\tau\rangle$ to investigate the vacuum quantum fluctuations of charge and its conjugate quantity for the system at finite temperature. For this purpose, we define the "coordinate" operator and the "momentum" operator in the fictitious space as follows

$$\tilde{\hat{q}} = \sqrt{\frac{\hbar}{2\mu\omega}}\left(\tilde{\hat{a}}^{\dagger} + \tilde{\hat{a}}\right), \qquad \tilde{\hat{p}} = i\sqrt{\frac{\mu\omega\hbar}{2}}\left(\tilde{\hat{a}}^{\dagger} - \tilde{\hat{a}}\right). \tag{18}$$

From (9), (13) and (18) we deduce

$$(\hat{q} - \tilde{\hat{q}})|\tau\rangle = \sqrt{\frac{2\hbar}{\mu\omega}}\tau_1|\tau\rangle, \tag{19}$$

$$(\hat{p} + \tilde{\hat{p}})|\tau\rangle = \sqrt{2\mu\omega\hbar}\,\tau_2|\tau\rangle, \tag{20}$$

which implies that $|\tau\rangle$ is the common eigenstate of the operators $\hat{q} - \tilde{\hat{q}}$ and $\hat{p} + \tilde{\hat{p}}$ ($[\hat{q} - \tilde{\hat{q}}, \hat{p} + \tilde{\hat{p}}] = 0$). From (15)–(16) we obtain

$$\hat{s}^{\dagger}(\theta)(\hat{q} - \tilde{\hat{q}})\hat{s}(\theta) = \frac{1}{\lambda}\int \frac{d^2\acute{\tau}}{\pi}|\tau'\rangle\langle\tau'/\lambda|(\hat{q} - \tilde{\hat{q}})\int \frac{d^2\tau}{\pi\lambda}|\tau/\lambda\rangle\langle\tau| = \frac{1}{\lambda}(\hat{q} - \tilde{\hat{q}}), \tag{21}$$

$$\hat{s}^{\dagger}(\theta)(\hat{p} + \tilde{\hat{p}})\hat{s}(\theta) = \frac{1}{\lambda}(\hat{p} + \tilde{\hat{p}}), \tag{22}$$

$$\hat{s}^{\dagger}(\theta)\hat{a}\hat{s}^{\dagger}(\theta) = \hat{a}\cosh\theta + \tilde{\hat{a}}^{\dagger}\sinh\theta, \qquad \hat{s}^{\dagger}(\theta)\tilde{\hat{a}}\hat{s}(\theta) = \tilde{\hat{a}}\cosh\theta + \hat{a}^{\dagger}\sinh\theta. \tag{23}$$

Thus the quantum fluctuations of $\hat{q} - \tilde{\hat{q}}$ and $\hat{p} + \tilde{\hat{p}}$ by using (21)–(23) are

$$\begin{aligned}\left\langle[\Delta(\hat{q} - \tilde{\hat{q}})]^2\right\rangle &= \langle 0\tilde{0}|\hat{s}^{\dagger}(\theta)(\hat{q} - \tilde{\hat{q}})^2\hat{s}(\theta)|0\tilde{0}\rangle - \left[\langle 0\tilde{0}|\hat{s}^{\dagger}(\theta)(\hat{q} - \tilde{\hat{q}})\hat{s}(\theta)|0\tilde{0}\rangle\right]^2 \\ &= \frac{\hbar}{\mu\omega}e^{-2\theta},\end{aligned} \tag{24}$$

$$\begin{aligned}\left\langle[\Delta(\hat{p}+\tilde{p})]^2\right\rangle &= \langle 0\tilde{0}|\hat{s}^\dagger(\theta)(\hat{p}+\tilde{p})^2\hat{s}(\theta)|0\tilde{0}\rangle - \left[\langle 0\tilde{0}|\hat{s}^\dagger(\theta)(\hat{p}+\tilde{p})\hat{s}(\theta)|0\tilde{0}\rangle\right]^2 \\ &= \mu\omega\hbar e^{-2\theta}. \end{aligned} \tag{25}$$

But we are interested in the quantum fluctuations of $\hat{q}$ and $\hat{p}$. In view of the following relations

$$\langle\hat{q}\tilde{q}\rangle = \langle 0\tilde{0}|\hat{s}^\dagger(\theta)\hat{q}\tilde{\tilde{q}}\hat{s}(\theta)|0\tilde{0}\rangle = \frac{\hbar}{2\mu\omega}\sinh 2\theta, \tag{26}$$

$$\langle\hat{p}\tilde{p}\rangle = \langle 0\tilde{0}|\hat{s}^\dagger(\theta)\hat{p}\tilde{\tilde{p}}\hat{s}(\theta)|0\tilde{0}\rangle = -\frac{\mu\omega\hbar}{2}\sinh 2\theta, \tag{27}$$

from (26)–(27) we have

$$\langle(\Delta\hat{q})^2\rangle = \langle 0\tilde{0}|\hat{s}^\dagger(\theta)\hat{q}^2\hat{s}(\theta)|0\tilde{0}\rangle = \frac{\hbar}{2\mu\omega}\cosh 2\theta, \tag{28}$$

$$\langle(\Delta\hat{p})^2\rangle = \langle 0\tilde{0}|\hat{s}^\dagger(\theta)\hat{p}^2\hat{s}(\theta)|0\tilde{0}\rangle = \frac{\mu\omega\hbar}{2}\cosh 2\theta. \tag{29}$$

So for the mesoscopic RLC circuit the quantum fluctuations of the charge and its conjugate quantity at finite temperature are respectively

$$\langle(\Delta\hat{q})^2\rangle = \frac{\hbar}{2}\sqrt{\frac{C}{L-2Rt}}\coth\left[\frac{\beta\hbar}{2}\sqrt{\frac{1}{(L-2Rt)C}}\right], \tag{30}$$

$$\langle(\Delta\hat{p})^2\rangle = \frac{\hbar}{2}\sqrt{\frac{L-2Rt}{C}}\coth\left[\frac{\beta\hbar}{2}\sqrt{\frac{1}{(L-2Rt)C}}\right]. \tag{31}$$

From (30)–(31) one can see the quantum fluctuations of the charge and its conjugate quantity depend on the time and temperature as well as the circuit cell parameters, as implies that the mesoscopic RLC circuit is a non-conservative system. The above results can be validated by another method named the generalized Hellmann-Feynman Theorem.

3.2 By Virtue of the Generalized Hellmann-Feynman Theorem

Reference [17] introduced the generalized Hellmann-Feynman Theorem

$$\frac{\partial\langle\hat{\mathcal{H}}(\varepsilon)\rangle_e}{\partial\varepsilon} = \frac{\partial\bar{E}(\varepsilon)}{\partial\varepsilon} = \left\langle\left[1+\beta\bar{E}(\varepsilon)-\beta\hat{\mathcal{H}}(\varepsilon)\right]\frac{\partial\hat{\mathcal{H}}(\varepsilon)}{\partial\varepsilon}\right\rangle_e, \tag{32}$$

where ε is a real parameter, $E(\varepsilon)$ is the eigenvalue of $\hat{\mathcal{H}}(\varepsilon)$ and “e” means ensemble average. When $\hat{\mathcal{H}}(\lambda)$ is independent of β, (32) can be rewritten as

$$\begin{aligned}\frac{\partial\bar{E}(\varepsilon)}{\partial\varepsilon} &= \left[1+\beta\bar{E}(\varepsilon)\right]\left\langle\frac{\partial\hat{\mathcal{H}}(\varepsilon))}{\partial\varepsilon}\right\rangle_e - \beta\left[-\frac{\partial}{\partial\beta}\left\langle\frac{\partial\hat{\mathcal{H}}(\varepsilon)}{\partial\varepsilon}\right\rangle_e + \left\langle\frac{\partial\hat{\mathcal{H}}(\varepsilon)}{\partial\varepsilon}\right\rangle_e\bar{E}(\varepsilon)\right] \\ &= \frac{\partial}{\partial\beta}\left[\beta\left\langle\frac{\partial\hat{\mathcal{H}}(\varepsilon)}{\partial\varepsilon}\right\rangle_e\right], \end{aligned} \tag{33}$$

whose integral form is

$$\beta\left\langle\frac{\partial\hat{\mathcal{H}}(\varepsilon)}{\partial\varepsilon}\right\rangle_e=\int d\beta\frac{\partial\bar{E}(\varepsilon)}{\partial\varepsilon},\tag{34}$$

where we have let the integral constant be zero. For the Hamilton operator given by (8), the ensemble average of energy is

$$\bar{E}(\varepsilon)=\langle\hat{\mathcal{H}}(\varepsilon)\rangle_e=\frac{Tr[e^{-\beta\hat{\mathcal{H}}(\varepsilon)}\hat{\mathcal{H}}(\varepsilon)]}{Tr[e^{-\beta\hat{\mathcal{H}}(\varepsilon)}]}=\frac{\hbar\omega}{2}\coth\frac{\hbar\omega\beta}{2}.\tag{35}$$

If at some certain time the capacitance C is referred to as parameter, we have

$$\begin{aligned}\beta\left\langle\frac{\partial\hat{\mathcal{H}}}{\partial C}\right\rangle_e&=-\frac{\beta}{2C^2}\langle\hat{q}^2\rangle_e=\int d\beta\frac{\partial}{\partial C}\left[\frac{\hbar\omega}{2}\coth\frac{\hbar\omega\beta}{2}\right]\\&=-\frac{\hbar\omega\beta}{4C}\coth\frac{\hbar\omega\beta}{2},\end{aligned}\tag{36}$$

i.e.,

$$\langle\hat{q}^2\rangle_e=\frac{C\hbar\omega}{2}\coth\frac{\hbar\omega\beta}{2}=\frac{\hbar}{2}\sqrt{\frac{C}{L+2Rt}}\coth\left[\frac{\beta\hbar}{2}\sqrt{\frac{1}{(L+2Rt)C}}\right].\tag{37}$$

Similarly, taking the inductance $L(t)$ as parameter, we obtain

$$\beta\left\langle\frac{\partial\hat{\mathcal{H}}}{\partial L(t)}\right\rangle_e=-\frac{\beta}{2L(t)^2}\langle\hat{p}^2\rangle_e=\int d\beta\frac{\partial}{\partial L(t)}\left[\frac{\hbar\omega}{2}\coth\frac{\hbar\omega\beta}{2}\right]=-\frac{\hbar\omega\beta}{4L(t)}\coth\frac{\hbar\omega\beta}{2},\tag{38}$$

i.e.,

$$\langle\hat{p}^2\rangle_e=\frac{L(t)\omega\hbar}{2}\coth\frac{\hbar\omega\beta}{2}=\frac{\hbar}{2}\sqrt{\frac{L-2Rt}{C}}\coth\left[\frac{\beta\hbar}{2}\sqrt{\frac{1}{(L-2Rt)C}}\right].\tag{39}$$

For $\langle\hat{p}\rangle_e=\langle\hat{q}\rangle_e=0$, we deduce

$$\left\langle(\Delta\hat{q})^2\right\rangle_e=\frac{\hbar}{2}\sqrt{\frac{C}{L-2Rt}}\coth\left[\frac{\beta\hbar}{2}\sqrt{\frac{1}{(L-2Rt)C}}\right],\tag{40}$$

$$\left\langle(\Delta\hat{p})^2\right\rangle_e=\frac{\hbar}{2}\sqrt{\frac{L-2Rt}{C}}\coth\left[\frac{\beta\hbar}{2}\sqrt{\frac{1}{(L-2Rt)C}}\right].\tag{41}$$

The above results are equivalent with (30)–(31), respectively. The difference between two methods only lies in that one is expectation values with a pure state, but another is ensemble averages. In addition, if $R\to 0$, from (40) and (41), we have

$$\left\langle(\Delta\hat{q})^2\right\rangle_e=\frac{\hbar}{2}\sqrt{\frac{C}{L}}\coth\left[\frac{\beta\hbar}{2}\sqrt{\frac{1}{LC}}\right],\tag{42}$$

$$\left\langle(\Delta\hat{p})^2\right\rangle_e=\frac{\hbar}{2}\sqrt{\frac{L}{C}}\coth\left[\frac{\beta\hbar}{2}\sqrt{\frac{1}{LC}}\right].\tag{43}$$

which are same as the results about indissipative mesoscopic LC circuit from the thermo field dynamics [18].

In summary, we quantize the mesoscopic RLC circuit by employing the canonical quantization method in which such a circuit can be equivalent to a changing mass simple harmonic oscillator. We believe that such a equivalent method is significant to dealing with other dissipative mesoscopic circuit. In the same time we discuss the thermal effect for the mesoscopic RLC circuit at finite temperature and explain how to introduce the thermal entangled state representation and the generalized Hellmann-Feynman Theorem into the investigation on thermal effect of the mesoscopic circuit.

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